Year 12 Mathematics

IAS 2.4

Trigonometric Relationships

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Contents

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NCEA 2 Internal Achievement Standard 2.4 – Trigonometric Relationships

This achievement standard involves applying trigonometric relationships in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply trigonometric relationships in solving problems.	• Apply trigonometric relationships, using relational thinking, in solving problems.	• Apply trigonometric relationships, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 7 of The New Zealand Curriculum, and is related to the achievement objective:
  - apply trigonometric relationships, including the sine and cosine rules, in two and three dimensions.

- Apply trigonometric relationships in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of trigonometric concepts and terms
  - communicating using appropriate representations.

- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

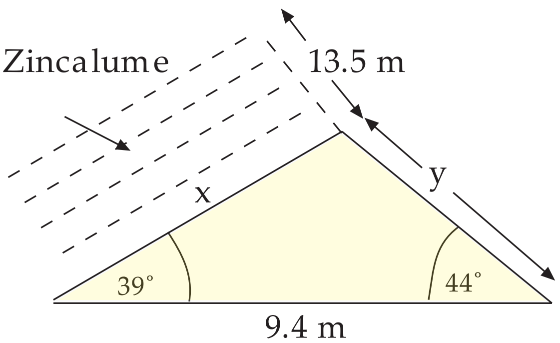
  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.

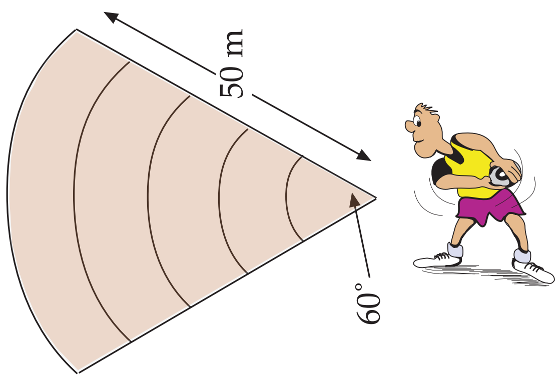
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.

- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

- Methods include a selection from those related to:
  - length of an arc of a circle
  - area of a sector of a circle
  - sine rule
  - cosine rule
  - area of a triangle.





Circular Measure

Radians

The convention to divide a circle into 360° appears to have been used from about 140 BC.

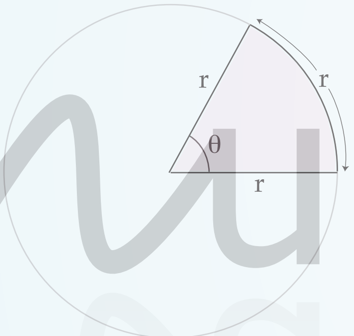
It probably began from dividing the 12 signs of the zodiac into smaller parts. A cycle of the seasons of approximately 360 days could be made to correspond to the 12 signs of the zodiac.

As convenient as this is, it does not give a mathematical basis for degrees.

Senior mathematics usually does not use degrees but defines another measure for angles called radians.

Radians at first appear clumsy (there are 6.2832 radians in a circle) but they make the solving of many trigonometric problems, and in particular calculus, easier.

A radian is defined as that angle whose arc is the same length as the radius of the arc.



An angle of one radian forms an arc length equal in length to the radius of the sector.

θ = one radian

A full circle has an arc length of $2\pi r$ and a radius of r so the number of radians in a circle is $\frac{2\pi r}{r}$ or 2π.

There are 2π or approximately 6.2832 radians in a circle so

$$2\pi \text{ radians} \equiv 360^\circ$$

approximately

$$6.2832 \text{ radians} \equiv 360^\circ$$

$$1 \text{ radian} \equiv \frac{360}{6.2832}$$

$$\equiv 57.3^\circ \qquad \text{(1 dp)}$$

The symbol ≡ means equivalent to. It is used when two quantities are the same but with different units.

Degrees to radian conversion is also covered in the Graphical Models Achievement Standard. If you have already done this there is no need to repeat it.

Conversions

Degrees to Radians

We use the exact conversion

$$2\pi \text{ rad.} \equiv 360^\circ$$

or

$$\pi \text{ rad.} \equiv 180^\circ$$

to convert from degrees to radians and vice versa.

To convert θ degrees to radians all we have to find is what fraction of 180° it is and multiply by π.

$$\text{Angle in radians} = \frac{\theta \times \pi}{180}$$

You can use the fraction button on your calculator to simplify this fraction quickly.

Radians to Degrees

To convert θ radians to degrees we divide by π and multiply by 180.

$$\text{Angle in degrees} = \frac{\theta \times 180}{\pi}$$

If you are given an angle in radians which includes the symbol π (e.g. $\frac{\pi}{6}$) it is easier to convert it to degrees by just replacing the π symbol with 180° (See the Example on the next page).

Example

Convert the angles in degrees to radians.

a) 71.4°

b) 135° leaving your answer in terms of π.

a)

$$\theta \text{ rad.} = \frac{\theta \times \pi}{180}$$

$$= \frac{71.4 \times \pi}{180}$$

$$= 1.25 \qquad \text{(2 dp)}$$

b)

$$= \frac{\theta \times \pi}{180}$$

$$= \frac{135 \times \pi}{180}$$

and simplify the fraction.

$$= \frac{3\pi}{4}$$

Merit – Answer the following.

41. A car has a weak spot on one of its tyres. It will eventually fail and you are to explore how many times the weak spot is in contact with the road.

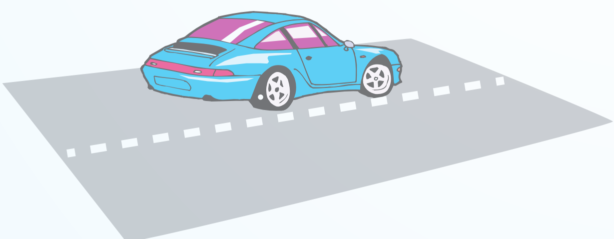
The car is travelling at an average speed of 90 km/h and each wheel is 37 cm in radius. You will need to consider the following:

- the speed of the car in m/s
- the distance travelled by the car in one revolution of the wheel
- the number of revolutions of the wheel in one second.

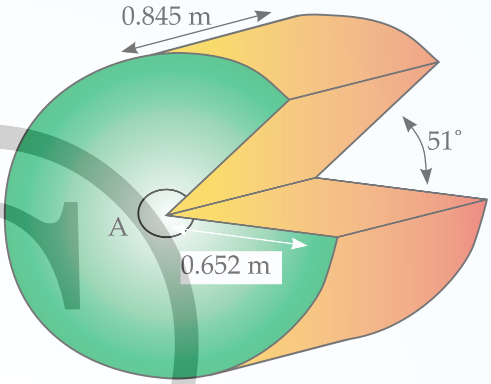
The car is being driven from Napier to Wellington which will take 3.75 hours.

How many times do you expect the weak spot to be in contact with the road?

Make sure you explain each step of your working when solving this problem.



42. A solid cylinder (radius 652 mm) has a wedge of 51.0° cut out of it so it sort of looks similar to a Pac-man.
The cylinder is 0.845 m in depth.



The side surface area (ignoring both ends) is to be covered in expensive gold foil. You will need to

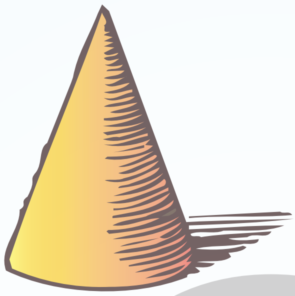
- find the green angle A in radians
- find the arc length of the green circular sector
- the total distance around the green circular sector
- the area of the gold foil.

The foil costs $75 a square metre. What is the estimated cost of the foil required?

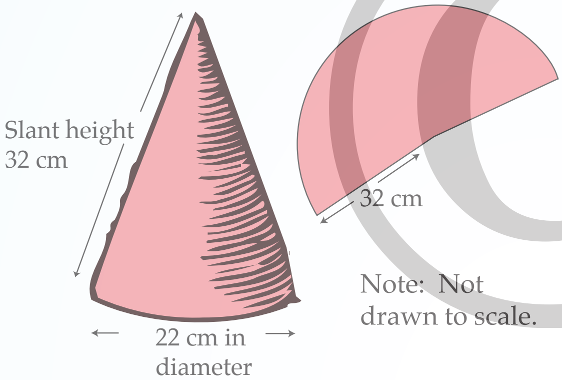
Make sure you explain each step of your working when solving this problem.

Excellence – Solve these more complex problems explaining what you are doing at each step.

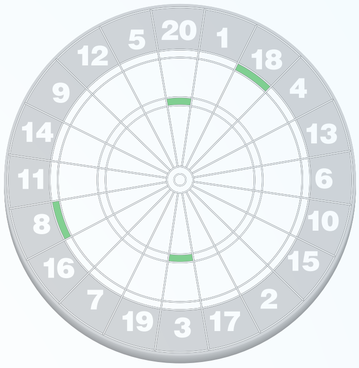
56. A conical hat is made by cutting a sector from a circle of card and joining the edges. The hat has a slant height of 32 cm and the resulting hat has a diameter of 22 cm.



Find the angle of the sector cut from the circle and the area of the resulting hat.

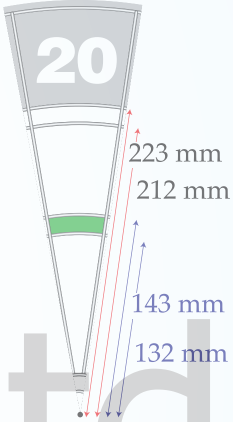


57. At a sideshow at an A & P Show there is a modified dart board with four sections shaded green.



The sideshow offers a $100 prize if one dart (costing $2) lands in any of the four green sections. You want to find the probability that by chance a dart could land in a winning area. The dart board has a radius of 312 mm and an area of 305 800 mm² (4 sf).

A close up of one of the sectors gives you the following dimensions. The inner green section (of which two are green) goes from 132 mm to 143 mm while the outer section (of which two are green) goes from 212 mm to 223 mm.



You will need to consider:

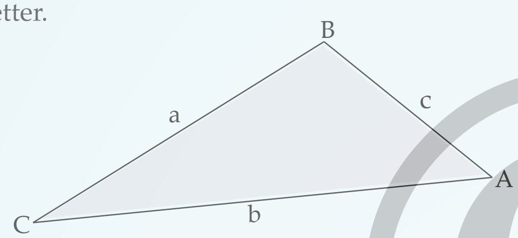
- the angular measure of each wedge
- the area of the two inner and outer green sections
- the ratio of the winning area to the total board area

in deciding whether this game offers a good random chance of winning.

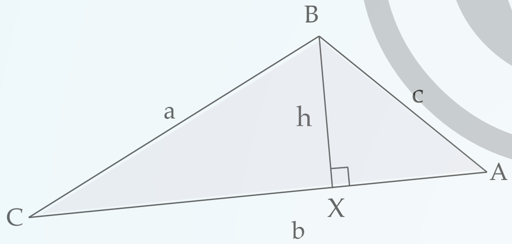
Sine Rule

Deriving the Sine Rule

The angles of a non-right-angled triangle are labelled with uppercase letters and each angle's opposite side with the corresponding lowercase letter.



To derive the sine rule an altitude is temporarily constructed to one of the sides.



From triangle BCX we find that

$$\sin C = \frac{h}{a}$$

$$h = a \sin C$$

and from triangle ABX we get

$$\sin A = \frac{h}{c}$$

$$h = c \sin A$$

Therefore

$$a \sin C = c \sin A$$

or $$\frac{a}{\sin A} = \frac{c}{\sin C}$$

A similar approach is used to extend the relation to side b and sin B.

Giving $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

The Sine Rule Formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b and c are the lengths of the three sides and A, B and C are the angles opposite each of the corresponding sides.

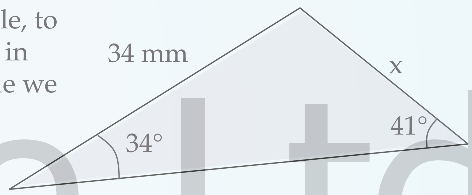
The sine rule can be expressed in two forms. The form below is useful when we are required to calculate an angle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

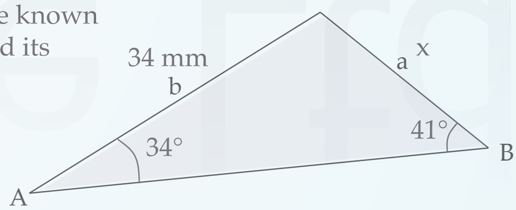
Only two of the three parts of these formulae are ever used to solve a single problem.

It is easier if we always represent our unknown with the letter a or if the unknown is an angle, the letter A.

For example, to calculate x in this triangle we start by labelling side x as a.



We then label its opposite angle A and the known side b and its angle B.



It does not matter how we label the other sides as long as we have sides opposite their corresponding angle.

You can use your graphics calculator for sine and cosine rule problems by entering the problem in the solver and getting the calculator to solve for the unknown.

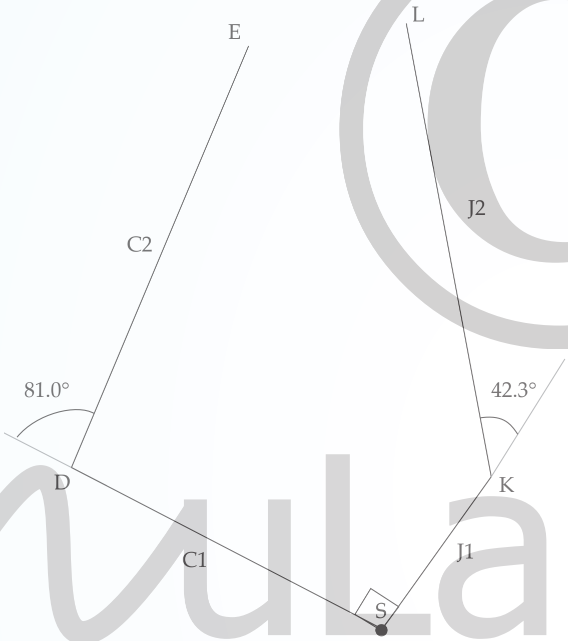
Extra Notes

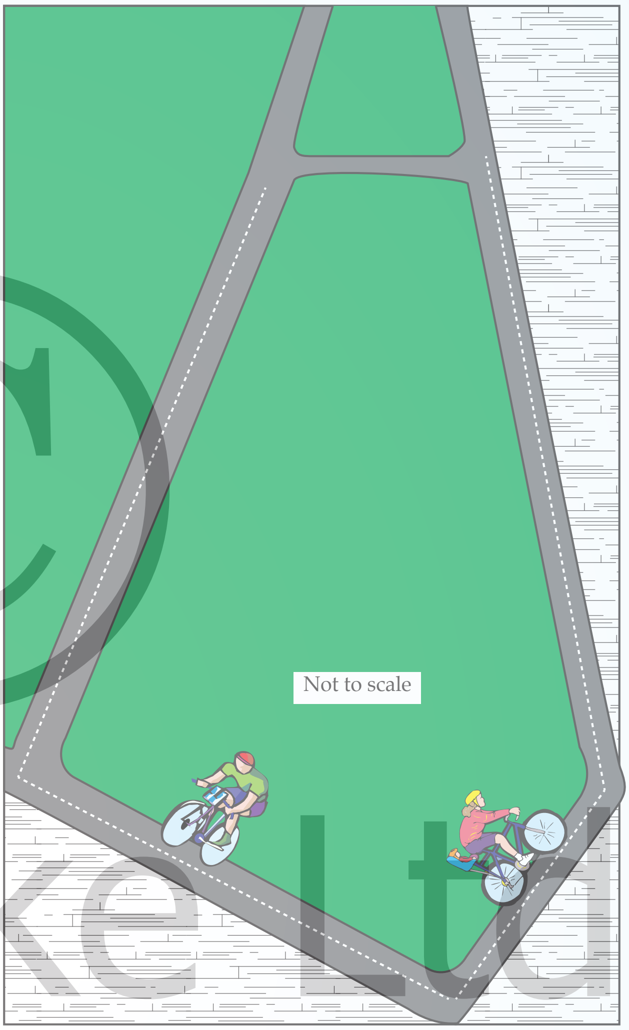
Excellence – Solve this trigonometric problem explaining what you are doing at each step.

117. Clint and Jenny enjoy long distance cycling. On a particular weekend they both head off at right angles to each other. They agree to stop and ring each other after about 75 minutes.

Clint cycles at 4.1 m/s and after 32.3 minutes must turn 81.0˚ right and then straight for 43.5 minutes.

Jenny (with Molly on the back) cycles straight at 3.9 m/s for 21.6 minutes then turns 42.3˚ left and continues for 54.8 minutes.





a) How far are Clint and Jenny from the start point when they stop?

b) How far are they from each other when they stop?

Practice Internal Assessment Task 2

Trigonometric Relationships 2.4

INTRODUCTION

The perimeter of a quadrilateral shaped fitness track at a primary school is marked out by the groundsman with four pegs A, B, C and D. A plan of the track is drawn below with all the boundary lengths and one angle marked. Each school level (junior, middle and senior) complete different paths around the pegs. The school would like to calculate the distances for each section and to also calculate the area enclosed by the pegs so they can order the correct amount of lawn fertiliser.

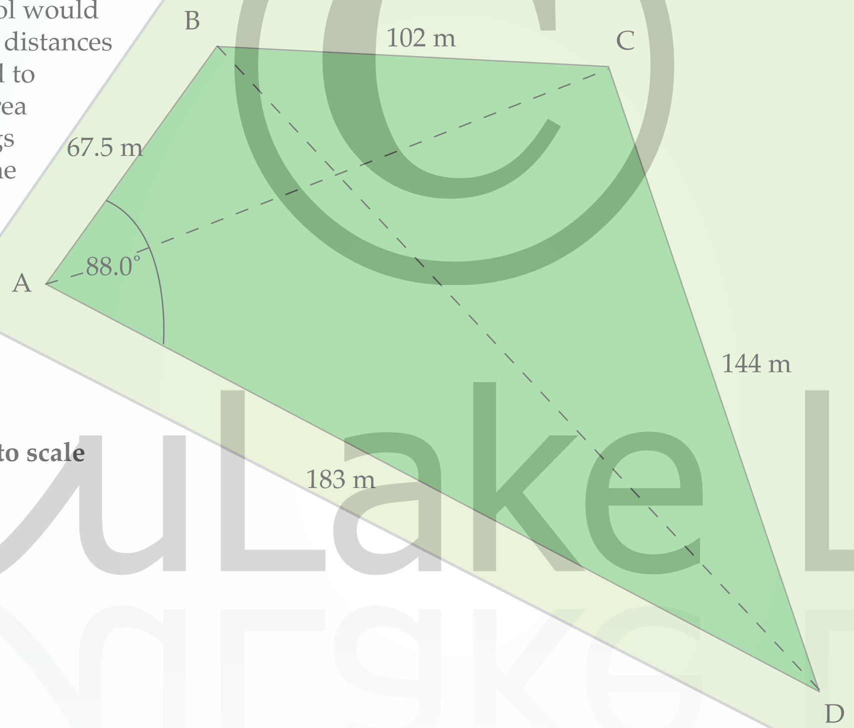


Diagram NOT to scale

The Year 7 and 8 students (seniors) run around ABCD (a distance of 496.5 m), the middle school around ABD and the juniors run around ABC. Working independently you are required to:

- find the distance the middle and junior school students run in one lap.
- calculate the area enclosed by the four pegs ABCD.

The staff would like to adjust peg B so the seniors complete a 500 m lap but they are unable to change angle BAD as it is the limited by the boundary fence of the school.

- Find a new position for peg B and the instructions that should be given to the groundsman.

The quality of your reasoning, using a range of methods, and how well you link this context to your solutions will determine your overall grade.

Clearly communicate your method using appropriate mathematical statements and working.

Answers

Page 4

1. 1.070 (3 dp)
2. 0.785 (3 dp)
3. 3.547 (3 dp)
4. 6.283 (3 dp)
5. 0.838 (3 dp)
6. 4.294 (3 dp)
7. 10.036 (3 dp)
8. 18.047 (3 dp)
9. $\frac{\pi}{2}$
10. π
11. $\frac{\pi}{6}$
12. $\frac{\pi}{12}$

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13. $\frac{\pi}{4}$
14. $\frac{7\pi}{6}$
15. $\frac{\pi}{3}$
16. $\frac{5\pi}{12}$
17. $\frac{3\pi}{10}$
18. $\frac{3\pi}{4}$
19. $\frac{26\pi}{9}$
20. $\frac{14\pi}{3}$
21. 114.6° (1 dp)
22. 31.0° (1 dp)
23. 120.0° (1 dp)
24. ⁻71.0° (1 dp)
25. 235.0° (1 dp)
26. 345.0° (1 dp)
27. 470.0° (1 dp)
28. 896.7° (1 dp)
29. 180°
30. 45°
31. 90°
32. 60°
33. 30°
34. 225°

Page 5 cont...

35. 240°
36. 270°
37. 540°
38. 210°
39. 80°
40. 162°

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41. The speed of the car in m/s
Speed = 90 000 / 3600
= 25 m/s
Circumference of wheel
C = 2π x 0.37
= 2.3248 m
Revolutions per second
Revs = 25 / 2.3248
= 10.754 revs / s
Time to Wellington
Time = 3.75 x 3600
= 13 500 s
Revolutions to Wellington
Revs = 13 500 x 10.754
= 145 200 (4 sf)
The weak spot will touch the road about 145 200 times (4 sf).

42. We need the angle in radians to calculate the arc length. Angle cut out
Angle out = 0.890
Angle A = 2π – 0.890
= 5.393 radians
Arc length of green
Arc = 5.393 x 0.652
= 3.5163 m (5 sf)
Perimeter + wedge
Dist. = 3.5163 + 2 x 0.652
= 4.8203 m (5 sf)
Area of gold
Area = 4.8203 x 0.845
= 4.0731 m² (5 sf)
Cost of foil
Cost = $305.49 (2 dp)

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43. 5.97 m (3 sf)
44. 1.235 radians (3 dp)
70.8° (1 dp)

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45. a = 6.3 m (2 sf)
46. 1.47 radians or 84.0° (3 sf)

Page 9 cont...

47. Arc = 5.52 m (3 sf)
Sector = 6.84 m² (3 sf)
48. Arc = 153 cm (3 sf)
Sector = 1940 cm² (3 sf)
49. Perimeter = 545 m (3 sf)
50. r = 9.80 m (2 sf)
51. 72 000 mm² (3 sf)
52. Arc = 3.29 m (3 sf)
Area = 2.39 m² (3 sf)

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53. a) 2.71 radians
b) 0.61 m² (2 sf)
54. Angle = 4.5 radians
or = 256.9°
Area = 6800 mm² (2 sf)
55. a) 1300 m² (2 sf)
b) 152 m
c) 105 m

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56. Let r = radius hat,
R = radius card.
θR = 2πr
32θ = 2π11
$\theta = \frac{11}{16}\pi$ (2.16)
cut out $= \frac{21}{16}\pi$ (4.12)
Area = 1100 cm²

57. Angle of one sector.
Angle = 0.314 16 rad.
Area in = 0.5π(143² – 132²)
= 475 mm² (0 dp)
Area out = 0.5π(223² – 212²)
= 752 mm² (0 dp)
Total winning area
= 2(475 + 752)
= 2454 mm²
Ratio win to total
= 305 800 : 2454
= 125 : 1
Therefore given odds of $100 : $2 or 50 : 1 are not good particularly as you may miss the entire board.

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58. Area = 32 m² (2 sf)
59. Area = 100 cm² (2 sf)
60. Area = 75.9 m² (1 dp)
61. Area = 427 cm² (3 sf)
62. Area = 26.8 m² (3 sf)